A Model for Pricing Derivatives on Ceiling Underlying Variables

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Abstract

Eurodollar futures, Euroyen futures, and EuroCanada futures represent financial assets which have ceilings. This paper presents a theory which establishes a risk neutral valuation relationship (RNVR) for pricing derivatives written on upper bounded underlying variables. First, the theory is developed in a single period economy. It is assumed that there is a representative agent with a particular utility function of the HARA family of utility functions, and that aggregate wealth and the underlying variable are bivariate upper bound or negatively skewed lognormally distributed. Second, the theory is developed in a continous-time framework where the risk aversion assumption is dropped, and replaced by the assumption of two long lived underlyings dynamically traded. It is assumed that the risky underlying follows an upper bound or negatively skew geometric brownian motion which has, at the end of each period, an upper bound lognormal distribution. The model is applied to derive closed-form solutions for the price of call and put options. These solutions depend on an extra parameter, not contained in the Black-Scholes model, the upper bound parameter.

Introduction

Black and Scholes (1973) and Merton (1973) establish the first risk neutral valuation relationship (RNVR)[1]. This RNVR is a relation between the value of a contingent claim and the value of the underlying variables, which depends on other exogenous parameters, but is compatible with arbitrary preference parameters, in particular with risk neutrality, under which all assets yield the same equilibrium expected rate of return. They assume that two assets are dynamically traded and, then, that the payoff of a contingent claim can be continuously replicated. In this Black-Scholes-Merton world risky assets follow a geometric brownian motion and then have a lognormal distribution at the end of each period.

A utility-based approach to the risk neutral valuation of contingent claims is addressed, among others, by Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam (1984), and Turnbull and Milne (1991)[2]. Under this approach, the Black-Scholes model may be derived, for example, assuming a representative agent whose utility function displays constant proportional risk aversion (CPRA). Aggregate wealth and the underlying risky variable are bivariate lognormally distributed.

The no-arbitrage approach to the risk neutral valuation of contingent claims is addressed, among others,[3] by Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981). Under this approach, it is shown that if there are no arbitrage opportunities then discounted prices are martingales under an equivalent probability measure. The assumption that agents are risk averse is dropped. It is assumed that markets are completed by dynamically trading two assets, a stock and a bond.

This paper aims at complementing the option pricing theory (OPT), deriving a risk neutral valuation relation for the pricing of contingent claims when the underlying variable is upper bounded. Such model might be useful to price derivatives when the underlying variables show behavior which fits reasonably in one of the following two cases:

First, in some cases, two financial variables are related by a straight line with a negative slope. In this case, if one variable has a lognormal distribution then the

[1]The expression risk neutral valuation relationship (RNVR) is due to Brennan (1979).

[2]See also the preference-based models derived by Camara (2003) and (2005) and Schroder (2004).

[3]See the review article by Cox and Huang (1989).

other has an upper bound or negatively skewed lognormal distribution [4]. As an example consider the interest rate derivatives market, where there are related price-based options and yield-based options. In some cases, the price of a financial contract is equal to a threshold less a yield or, if one prefers, the yield is equal to the threshold less the price. The Chicago Mercantile Exchange (CME) lists options on treasury bill futures, Eurodollar futures, EuroCanada futures, Euroyen futures, and LIBOR futures. The valuation of options on Eurodollars must be consistent with the pricing of options on its implied interest rate, since a Eurocontract futures price is equal to 100 less the implied interest rate. In the same way, the valuation of the price-based options on 13-week T-bill futures listed at the CME must be consistent with the valuation of the yield-based options on 13-week T-bills listed at the Chicago Board Options Exchange (CBOE). Suppose that the Black (1976) model is used by a bond portfolio manager to price derivatives written on the price, then the manager should use the upper bound lognormal risk neutral option pricing model [5] to get values for yield-based options.

Second, some underlying variables might have negative values. In general, cash-flows which enter into real option problems might have both inflows and outflows. For example, the operating cash-flow is the fundamental underlying variable of many real option problems, such as the option to defer investment, the option to expand the scale of production, or the option to abandon a project permanently. Some exotic spread options require an underlying stochastic variable that might assume both negative and positive values [6]. The Black-Scholes model cannot be used to price contingent claims on underlying variables which might have negative values, since it has an implied (risk-neutral) lognormal probability density function (PDF). Therefore, the true or actual distribution of the underlying variable also gives a probability of zero to the event in which the underlying variable has negative values. The negatively skewed or upper bound lognormal gives a positive probability to both events, inflows and outflows.

The main goal of this article is to establish a theory for the risk neutral pricing of derivatives when the underlying variable behaves as described before. First, the theory is developed in a single period discrete-time economy. Second, the theory is developed in a multi-period continuous-time economy to consistently price options with different maturities in the same economy [7].

In the first part of the article it is assumed that markets open at the beginning and the end of the economy, and then that there is no trade between those two dates. The set of securities in the economy is countably finite. It is assumed that the aggregation problem [8] is already solved, which is a sufficient condition for prices to be derived in a representative agent economy.

The article establishes a risk neutral valuation relationship (RNVR) assuming both that there is a representative agent with a particular utility function of the HARA family of utility functions and that aggregate wealth and the underlying risky variable are bivariate upper bound or negatively skewed lognormally distributed. Therefore, markets are dynamically incomplete. The assumption of a particular type of absolute risk aversion is an extremely strong one, but should only be seen as a sufficient condition to derive the option pricing formulae. The paper derives a RNVR, that might be useful to price real options, without the double assumption that it is possible to construct and

[4] Aitchison and Brown (1957) p.16 define $x = \theta - y$, where x has a lognormal distribution and θ is the upper bound of the negatively skew or upper bound lognormal random variable y. It should be noted that any straight line with an arbitrary negative slope and x lognormal might be expressed in that form.

[5] The model derived in this article might be easily extended to price interest-rate derivatives in the same form that the Black (1976) model extends the Black-Scholes (1973) model.

[6] See Zhang (1998), p.491.

[7] The motivation for this developement of the theory arises from the fact that it is not possible to price consistently options with different maturities in the same single period economy.

[8] The aggregation property is solved when prices in the economy are determined independently of the distribution of initial wealth. Rubinstein (1974) and Brennan and Kraus (1976) derive, respectively, sufficient and necessary conditions for aggregation. These conditions are, in the case of the utility function used in this article, that all investors have identical cautiousness.

maintain a replicating portfolio and that there is continuous trading. The RNVR allows one to price any contingent claim using the negatively skewed or upper bound lognormal risk neutral PDF.

The risk neutral valuation relationship is applied to derive closed-form solutions for both the price of a call option and the price of a put option. It is shown that there is a relation between the price of a negatively skewed or upper bound call (put) and the value of a lognormal or Black-Scholes put (call). As an implication of this fact, the paper gives conditions for the value of a portfolio of call options to be equal to the price of a portfolio of put options.

The following part of the paper drops the risk aversion assumption. There are two assets in this economy, a risky asset and a riskless one. The price of the risky asset is governed by an upper bound or negatively skewed geometric brownian motion. The solution of the negatively skewed geometric brownian motion has, at the end of each period, a negatively skewed lognormal distribution. Assuming that it is possible to replicate the payoff value of a contingent claim throught continuous trading, the paper derives an RNVR in continuoustime. This explores the martingale approach of Harrison and Kreps (1979) and Harrison and Pliska (1981).

The issue of an upper bounded underlying variable has been addressed in the option pricing literature, but in distinguishable terms of the present paper, among others, by Brennan and Schwartz (1985), Dixit (1989), Stapleton and Subrahmanyam (1993), Dixit and Pindyck (1994), and Trigeorgis (1996). All this work, with the exception of the paper by Stapleton and Subrahmanyam (1993), contributes to the real options literature. In general, such research rests on the explicit assumption that replicating portfolios may be formed by continuously trading in the futures contracts of the commodity [9] or that there is a risk neutral firm [10]. In general, to use upper bounds, the researchers have truncated the processes followed by the variables at the threshold level. This contrasts with the model derived in this paper, where the variable is not truncated, but its density has an upper bound in the same natural way that the lognormal has a lower bound at zero. Stapleton and Subrahmanyam (1993) develop a binomial model to price interest-rate options for when the underlying has a bound at the unit. This contrasts with our continuous state-space model with an arbitrary threshold parameter.

The remainder of the paper is organized as follows. Section 2 explains what is option pricing theory, and it is directed to those readers who do not have a finance background. Section 3 presents the discrete time model. Section 4 presents the continuous time model. Section 5 concludes.

Option Pricing Theory

The modern option pricing theory begins with the work by Black and Scholes (1973) and Merton (1973), who derived the first preference-free closed-form solution for the price of a stock option [11]. According to the Black-Scholes valuation equation, the price of the option depends on five variables: the current stock price, the strike or exercise price, the maturity of the option, the riskless interest rate, and the volatility or standard deviation of stock returns. The first four variables are observable, while the volatility of the stock is relatively easy to estimate. Hence, in order to use the formula, we first obtain the values of these five variables, then we plug them into the Black-Scholes valuation equation, and as result we obtain the price of the option.

The Black-Scholes valuation equation is valuable because:

1. It does not depend on preference parameters. This result is important since preference parameters are very difficult to estimate. The option pricing equations derived previously to the Black-Scholes (1973) work depended either on preferences or other arbitrary parameters that are very difficult to obtain.

[9] See Brennan and Schwartz (1985).

[10] See Dixit (1989).

[11] This section presents a minor extension of the overview of option pricing theory given by Camara (2004).

2. It does not depend on the location parameter of the stock price distribution (i.e. the actual expected return under their assumption that stock returns are normally distributed). This is relevant since the location parameter is, in practice, very difficult to estimate with precision.

3. It is compatible with risk-neutrality; i.e. a world where all assets yield the riskless rate of return. No risk-premium affects the equation. This is important since risk-premiums are also, in practice, very difficult to estimate.

4. It is obtained under no-arbitrage conditions and, therefore, sustained by some equilibrium economy.

5. It is obtained in closed-form and, therefore, easily applied in practice.

The first of these five characteristics is not a surprising result given the assumptions of Black and Scholes (1973). These authors assume that there are no arbitrage opportunities in the economy, and that the stock and options written on the stock might be continuously traded. Under these assumptions it is possible to construct and to maintain a riskless portfolio (involving the stock and the option) which, since it is riskless, yields the riskless rate of return. From a mathematical point of view, this is a partial differential equation whose solution is the Black-Scholes valuation equation. Since preference parameters do not enter into the problem, it is not surprising that preferences do not affect the price of stock options.

Since its early stages the Black-Scholes model received a great deal of attention from academics and practitioners. One branch of the literature has investigated the investors attitudes toward risk and, in particular, the type of risk aversion that can sustain the Black- Scholes formulae in the pricing of stock options. This branch of research is interesting because it shows conditions on preferences and distributions that lead to the Black-Scholes option price when it is not possible to construct and to maintain a dynamic riskless portfolio. It is costly to trade dynamically stocks and options written on the stock and, therefore, it is also important from a practical point of view to know that the Black-Scholes equation holds under alternative assumptions to the dynamic riskless hedge assumption.

The earlier literature was almost unanimous in relating a power utility function with the Black-Scholes valuation model. Under such utility function, which displays constant proportional risk aversion (CPRA), the percentage invested in risky assets is unchanged as the wealth of the investors increases. For example, Rubinstein (1976) and Brennan (1979) remark that the Black-Scholes model can be obtained in an equilibrium economy, when agents have power utility functions characterized by CPRA, and aggregate wealth and the stock price are jointly lognormally distributed. Unfortunately, empirical and theoretical research has cast doubts on the reasonability of the CPRA assumption. There are many authors who believe that investors, instead of CPRA, display other types of preferences.

Recently, Camara (2003) and Schroder (2004) showed that the Black-Scholes valuation equation also holds with many other types of utility functions or preference functions. Therefore, Camara (2003) and Shroder (2004) derive many equilibrium economies that sustain the Black-Scholes valuation equation even if it is not possible to construct and to maintain a riskless portfolio. The practical implication of this is that, even if dynamic trading is not possible, there are still many situations were we can use the Black-Scholes valuation equation. Here, we extend this idea to other distributions.

There are other interesting extensions of the Black-Scholes work by those who study call and put options. A branch of the literature has investigated how to price other options besides call and put options. These are known as exotic options. Another branch of the literature has tested the implications of the Black-Scholes model, and has found that this model does not price options correctly in the marketplace. This has lead to the proposal of other models that, for example, include jumps in the stock price.

The One-Period Framework

The Economy

The model assumes that markets open at the beginning and the end of the economy, and then that there is no trade between those two dates. In such a situation a riskless hedge is not possible to construct and maintain, and dynamic trading does not exist. To price contingent claims in this economy, it is necessary to use a preference-based model.

The analysis starts by assuming that the initial consumption decision has already been taken. The representative investor maximizes the expected utility of end-of-period wealth, $E^P[U(W_1)]$, where:

$$W_1 = W_0 r + \sum_k \sum_j n_{kj} \left[f_k(S_{j1}, 1) - f_k(S_{j0}, 0) r \right]$$

P is the actual probability measure;

$U(.)$ is her utility for end-of-period wealth;

W_0 is the investor's initial wealth;

r is the riskless return, i.e. one plus the riskless rate of return; [12]

n_{kj} is the investor's demand for units of claims ($k = 1, \ldots, K$ for all j);

$f_k(S_{j1}, 1)$ is the end-of-period payoff associated with the contingent claim k as a function of the underlying variable j;

$f_k(S_{j0}, 0)$ is the current price of the claim $f_k(S_{j1}, 1)$;

S_{j1} is the end-of-period value of the underlying variable j.

Assuming that the representative agent is nonsatiated and risk averse, the expected utility is maximized when:

$$E^P[U'(W_1) f_k(S_{j1}, 1)] = r f_k(S_{j0}, 0) E^P[U'(W_1)]$$

or
$$f_k(S_{j0}, 0) = r^{-1} \frac{E^P[U'(W_1) f_k(S_{j1}, 1)]}{E^P[U'(W_1)]}$$

for all claims.

The current price of individual claims can also be written, using the law of iterated expectations, in terms of their end-of-period payoffs as follows:

$$f_k(S_{j0}, 0) = r^{-1} E^P[f_k(S_{j1}, 1) Z(S_{j1})] \quad (1)$$

where
$$Z(S_{j1}) = \frac{E^P[U'(W_1)|S_{j1}]}{E^P[U'(W_1)]} \quad (2)$$

defines the pricing kernel. In particular, the underlying risks S [13] can themselves be priced by using a particular application of the general formula,

$$S_0 = r^{-1} E^P[S Z(S)] \quad (3)$$

where S_0 is the current value of the underlying stochastic variable.

The general valuation formulae (1) and (3) will be used to investigate option prices in an economy where the representative agent has a particular utility function of the HARA family of utility functions and the marginal distribution of S is a negatively skewed lognormal.

The Negatively Skewed Lognormal and the Utility Function

Definition 1. (The bivariate negatively skewed lognormal distribution)

Let the two-dimensional random variable (W, S) have the joint probability density function (p.d.f.):

$$h(W, s) = \frac{1}{2\pi\sigma_w\sigma\sqrt{1-\rho^2}(\theta - S)(\tau - W)}$$
$$exp\left[-\frac{1}{2(1-\rho^2)} \left[\left(\frac{ln(\theta - S) - \mu}{\sigma} \right)^2 - 2\rho \frac{ln(\theta - S) - \mu}{\sigma} \frac{ln(\tau - W) - \mu_w}{\sigma_w} + \left(\frac{ln(\tau - W) - \mu_w}{\sigma_w} \right)^2 \right] \right] \quad (4)$$

for $W < \tau$, $S < \theta$, where σ_w, σ, μ_w, μ, ρ, θ are constants such that $-1 < \rho < 1$, $0 < \sigma_w$, $0 < \sigma$,

[12] Section 3 develops the theory in an intertemporal economy, where r will be the continuously compounded riskless rate of return.
[13] The subscripts k, j, and 1 will be dropped henceforth.

$-\infty < \mu_w < \infty$, $-\infty < \mu < \infty$, $-\infty < \theta < \infty$, and $-\infty < \tau < \infty$. Then the random variable (W, S) is defined to have a bivariate negatively skewed or upper bound lognormal distribution.

Lemma A2 of Appendix A shows that the actual marginal distribution of the underlying is a negatively skewed or upper bound lognormal $\Lambda^P(\mu, \sigma, \theta)$.

Definition 2. (The negatively skewed lognormal risk neutral density)

If $h(S)$ is the upper bound lognormal actual density of the underlying variable with parameters μ, σ, and θ i.e. $h(S)$ is $\Lambda^P(\mu, \sigma, \theta)$, then $g(S)$ is the negatively skew lognormal risk neutral density with scale parameter $ln(\theta - S_0 r) - \frac{1}{2}\sigma^2$, shape parameter σ, and threshold parameter θ i.e. $g(S)$ is $\Lambda^Q(ln(\theta - S_0 r) - \frac{1}{2}\sigma^2, \sigma, \theta)$ [14] where Q denotes the risk neutral or equivalent probability measure of the underlying risk.

Definition 3. (The marginal utility function)

The marginal utility function of end-of-period wealth is:

$$U'(W) = (\tau - W)^{\varphi} \tag{5}$$

where the $\tau > W > 0$ is required for nonsatiation and $\varphi > 0$ is required for risk aversion. [15]

Lemma 1.

Suppose that the representative agent has a marginal utility function given by equation (5). Let aggregate wealth and the underlying variable have a bivariate negatively skew lognormal distribution and, in particular, a bivariate p.d.f. given by equation (4). Then the pricing kernel $Z(S)$, defined by equation (2), is given by the following equation:

$$Z(S) = \frac{E^P[U'(W)|S]}{E^P[U'(W)]}$$
$$= exp\left[\rho\varphi\frac{\sigma_w}{\sigma}[ln(\theta - S) - \mu] - \frac{1}{2}\varphi^2\sigma_w^2\rho^2\right] \tag{6}$$

Proof: By lemma A1 of Appendix A, the marginal distribution of wealth is a univariate negatively skew lognormal. Using both lemma A1 and the definition of the φ moment about τ of a negatively skew lognormal random variable yields:

$$E^P[U'(W)] = exp\left[\varphi\mu_w + \frac{1}{2}\varphi^2\sigma^2\right]$$

By lemma A3 of Appendix A, the conditional distribution of wealth given the value of the underlying is a univariate negatively skew lognormal. Using both lemma A3 and the definition of the φ. moment about t of a negatively skew lognormal random variable yields:

$$E^P[U'(W)|S] = exp\left[\varphi\mu_w + \rho\varphi\frac{\sigma_w}{\sigma}[ln(\theta - S) - \mu] + \frac{1}{2}\varphi^2\sigma_w^2\left(1 - \rho^2\right)\right]$$

Substituting the last two equations in equation (2) yields the pricing kernel as given by equation (6).

Option Valuation

This subsection establishes a RNVR when the underlying stochastic variable has a marginal upper bound or negatively skewed lognormal distribution.

Proposition 1. (The discrete-time model)

Suppose that the representative agent has a marginal utility function given by equation (5). Let aggregate wealth and the underlying variable have a bivariate

[14] The moment about θ of $\Lambda^P(\mu, \sigma, \theta)$ is $E^P[(\theta - S)^\alpha] = exp[\alpha\mu + \frac{1}{2}\alpha^2\sigma^2]$. Therefore $E^P[S] = \theta - exp[\mu + \frac{1}{2}\sigma^2]$.

[15] The underlying utility function appears to be a power one, but it is a quadratic one. The utility function is given by $U(W) = \frac{1}{\varphi+1}(\tau - W)^{\varphi+1}$. Then the marginal utility function is given by equation (5), with the restriction $\tau > W > 0$ denoting nonsatiation. Also, $U''(W) = -\varphi(\tau - W)^{\varphi-1}$, with the restriction $\varphi > 0$ denoting risk aversion. Then the measure of absolute risk aversion $A(W) = -\frac{U''(W)}{U'(W)}$ is given by $A(W) = \psi(\tau - W)^{-1}$. Hence $A'(W) = \varphi(\tau - W)^{-2} > 0$, which shows that the utility function displays increasing absolute risk aversion.

negatively skewed lognormal distribution and, in particular, a bivariate p.d.f. given by equation (4). Then there is a general risk neutral valuation relationship for the pricing of contingent claims, i.e.

$$E^P[f_k(S_{j1},1)Z(S_{j1})] = E^Q[f_k(S_{j1},1)]$$

Proof: Suppose that the representative agent has a marginal utility function of wealth given by equation (5) and that the joint density of wealth and underlying variable is an upper bound lognormal given by equation (4). Then by lemma A2 of Appendix A, the marginal distribution of the underlying variable is an upper bound lognormal.

The current value of the underlying stochastic variable (3), which has a marginal upper bound lognormal distribution, after using the derived pricing kernel (6) and simplifying the resulting expression, is given by:

$$S_0\tau = \int_{-\infty}^{\infty} S \frac{1}{\sqrt{2\pi}\sigma(\theta - S)}$$
$$exp\left[-\frac{1}{2\sigma^2}\left[ln(\theta - S) - (\mu + \alpha\sigma_w\rho\sigma)\right]^2\right] dS \quad (7)$$

It follows directly, when the expression (7) is evaluated, that:

$$\alpha\sigma_w\rho\sigma + \mu = ln(\theta - rS_0) - \frac{\sigma^2}{2} \quad (8)$$

The current value of the contingent claim (1), since the underlying stochastic variable has a marginal negatively skewed lognormal distribution, after using the derived pricing kernel (6) and simplifying the resulting expression, can be written as:

$$f(S_0,0)r = \int_{-\infty}^{\infty} f(S,1)\frac{1}{\sqrt{2\pi}\sigma(\theta - S)}$$
$$exp\left[-\frac{1}{2\sigma^2}\left[ln(\theta - S) - (\mu + \alpha\sigma_w\rho\sigma)\right]^2\right] dS \quad (9)$$

Using the valuation relation (8) to eliminate preference parameters yields:

$$f(S_0,0) = \int_{-\infty}^{\infty} f(S,1)\frac{1}{\sqrt{2\pi}\sigma(\theta - S)}$$
$$exp\left[-\frac{1}{2\sigma^2}\left[ln(\theta - S) - (ln(\theta - S_0 r) - \frac{1}{2}\sigma^2)\right]^2\right] dS \quad (10)$$

which is a risk neutral valuation relationship (RNVR), since it satisfies definition 2.

Corollary 1. (Call and put option prices)

Suppose that the representative agent has a marginal utility function given by equation (5). Let aggregate wealth and the underlying variable have a bivariate upper bound lognormal distribution and, in particular, a bivariate p.d.f. given by equation (4). Then:

(i) the price of an European call option written on a nondividend paying underlying is given by the formula:

$$f_{0,c} = \left(S_0 - \theta r^{-1}\right) N(d_1) - r^{-1}(K - \theta)N(d_2) \quad (11)$$

(ii) the price of an European put option written on a nondividend paying underlying is given by the formula:

$$f_{0,p} = (K - \theta)r^{-1}N(-d_2) - \left(S_0 - \theta r^{-1}\right) N(-d_1) \quad (12)$$

where:

$$d_1 = \frac{ln\left(\frac{\theta - K}{\theta - S_0 r}\right) - \frac{1}{2}\sigma^2}{\sigma}$$

$$d_2 = d_1 + \sigma$$

where K is the exercise price and $N(.)$ is the cumulative distribution function of a standard normal random variable.

The next corollary relates Black-Scholes option values to negatively skewed option values, i.e. to option values whose underlying has a negatively skewed lognormal distribution.

Corollary 2. (Two put-call parities)

Suppose that the representative agent has a marginal utility function given by equation (5). Let aggregate wealth and the underlying variable have a bivariate negatively skewed lognormal distribution and, in particular, a bivariate p.d.f. given by (4). Let both

$S^* = \theta - S$ and $K^* = \theta - K$. Then there are two put-call parities:

$$f_{0,c} = f_{0,p}^{B \cdot S} \tag{13}$$

$$f_{0,p} = f_{0,c}^{B \cdot S} \tag{14}$$

where:

$f_{0,c}^{B \cdot S}$ is the Black-Scholes value for a call with strike price K^*;

$f_{0,p}^{B \cdot S}$ is the Black-Scholes value for a put with strike price K^*;

Proof: Suppose that the representative agent has a marginal utility function of wealth given by equation (5) and that the joint density of wealth and underlying variable is a negatively skew lognormal given by equation (4). Then by lemma A2 of Appendix A, the marginal distribution of the underlying variable S is an upper bound lognormal. If $S \sim \Lambda^P(\mu, \sigma, \theta)$ then $S^* \sim \Lambda^P(\mu, \sigma)$, by the relation between the lognormal distribution and the negatively skewed lognormal distribution. One can easily see that:

(i) $Max[S - K, 0] = Max[K^* - S^*, 0]$

(ii) $Max[K - S, 0] = Max[S^* - K^*, 0]$

where $K^* = \theta - K$. In words:

(i) A negatively skewed call option with a strike price K is equivalent to a Black-Scholes put option with a strike price K^*;

(ii) A negatively skewed put option with a strike price K is equivalent to a Black-Scholes call option with a strike price K^*;

Hence one can write equations (13) and (14) immediatly.

An immediate implication of the previous corollary is that, under the conditions of the corollary, there is a portfolio of calls equal to a portfolio of puts, and in particular:

$$f_{0,c} + f_{0,c}^{B \cdot S} = f_{0,p} + f_{0,p}^{B \cdot S} \tag{15}$$

Equation (15) shows conditions for the value of a portfolio of calls to be equal to the value of a portfolio of puts. The left-hand side of equation (15) is the value of a portfolio with two call options, one with a negatively skewed lognormal underlying S and exercise price K, and the other with a lognormal underlying S^* and exercise price K^*. The right-hand side of equation (15) is the value of a portfolio with two put options, one with a negatively skewed lognormal underlying S and exercise price K, and the other with a lognormal underlying S^* and exercise price K^*. Equations (13), (14) and (15) may have some practical interest for the interest-rate derivatives market, where there are related yield-based options and price-based options.

The Continuous-Time Framework

The Economy

In the following B is a one-dimensional Brownian motion, starting from the origin, defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\mathcal{F}$ is the standard filtration generated by B.

There are two assets in this economy, a risky asset and a riskless asset. The price of the risky asset S_t at time t is governed by the stochastic differential equation:

$$dS_t = \left[\mu S_t + r\theta e^{-r(T-t)} - \mu\theta e^{-r(T-t)}\right] dt - \sigma\left[\theta e^{-r(T-t)} - S_t\right] dB_t \tag{16}$$

where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \theta < \infty$, $r > 0$, and $\theta e^{-\tau T} > S_0$ are constants, and $t \in E[0, T]$. Equation (16) is the formula of an upper bound or negatively skewed geometric Brownian motion.

The explicit solution to the Stochastic Differential Equation (SDE) (16) with $t = T$ is given by the following equation:

$$S_T = \theta - \left(\theta e^{-rT} - S_0\right) e^{(\mu - \frac{1}{2}\sigma^2)T + B_T} \tag{17}$$

The proof of this statement is given in appendix B.

If the risky underlying follows a SDE given by equation (16) then S_T has a negatively skewed lognormal distribution:

$$S_T \sim \Lambda^P\left[ln\left(\theta e^{-rT} - S_0\right) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}, \theta\right] \tag{18}$$

To see this, one should note from equation (17) that S_T has a negatively skewed lognormal distribution with threshold parameter θ. The corresponding normally distributed random variable z_T is:

$$z_T = ln[\theta - S_T]$$

with $$E^P[z_T] = ln(\theta e^{-rT} - S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T$$

and $$Var^P[z_T] = \sigma^2 T$$

The price of the riskless asset β_t at time t is governed by the equation:

$$\beta_t = \beta_0 e^{rt} \tag{19}$$

where $\beta_0 > 0$

This section drops the assumption that the investor is risk averse. It is assumed that a nonsatiated investor trades dynamically two long lived securities to achieve an optimal random wealth at time T.

Option Valuation

Harrison and Kreps (1979) and Harrison and Pliska (1981) show that if the price system (S_t, β_t) with $t \in E[0, T]$ has no arbitrage opportunities then discounted prices are martingales under a equivalent martingale measure. We explore now this notion in the negatively skewed geometric Brownian motion setting to derive the pricing formulae for option prices.

Now, we change the measure of the underlying to derive closed-form solutions for option prices. Let Q be the measure on $\mathcal{F}$ that is equivalent to P and whose Radon-Nikodym derivative is given by:

$$\frac{dQ}{dP} = exp\left[\frac{r-\mu}{\sigma}B_t - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right]$$

Then: $$\tilde{B}_t = B_t + \frac{\mu - r}{\sigma}t$$

where $\left\{\tilde{B}_t, t \in E[0, T]\right\}$ is a Brownian motion on $[0, T]$ under Q. Equation (16) implies that we have Q-a.s.:

$$dS_t = rS_t dt - \sigma\left[\theta e^{-r(T-t)} - S_t\right]d\tilde{B}_t$$

or

$$S_t = S_0 + r\int_0^t S_u du - \sigma\int_0^t \left[\theta e^{-r(T-u)} - S_u\right]d\tilde{B}_u \tag{20}$$

with $r > 0$, $\sigma > 0$, and $\theta e^{-rT} > S_0$.

The solution to equation (20) with $t = T$ is Q-a.s. given by

$$S_T = \theta - \left(\theta e^{-rT} - S_0\right)e^{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\tilde{B}_T} \tag{21}$$

Therefore, S_T has Q-a.s. a negatively skew lognormal distribution:

$$\Lambda^Q\left[ln\left(\theta - S_0 e^{rT}\right) - \frac{1}{2}\sigma^2 T, \sigma\sqrt{T}, \theta\right] \tag{22}$$

To obtain the next equation, the martingale pricing formula (23), it was only necessary to show the existence of a unique equivalent martingale measure, which implies market completeness, due to Harrison and Pliska (1981). All contingent claims can then be dynamically replicated and priced by:

$$f(S_0, 0) = E^Q\left[e^{-rT}f(S_T, T)\right] \tag{23}$$

We can now use formulae (22) and (23) to price consistently derivatives of different maturities in the same economy.

Conclusion

This paper complements the option pricing theory, presenting a theory for pricing options when the underlying variable has an arbitrary upper bound. The theory is first derived in an utility-based economy similar to that of Brennan (1979). Then the theory is presented in a no-arbitrage economy following Merton (1973) and Harrison and Pliska (1981). The general risk neutral valuation relationship is applied to derive equations for the price of a call option and a put option. However, the risk neutral valuation relation may be used to price any contingent claims on ceiling underlyings.

This work might be extended in several directions. For example, if there are multiple underlying variables one might extend the multivariate RNVR of Stapleton and Subrahmanyam (1984). A continuous-time multivariate model can also be derived. The general theory derived in this paper might also be applied to investigate problems in the real options area.

The modern option pricing theory was initiated by Black and Scholes (1973). These authors derive a valuation equation for the price of call and put options which does not depend on preferences. The theory has implications for many areas including the valuation of corporate securities, real options, and executive compensation. For example, the valuation of certain bonds can be done using an extension of the model proposed by Black-Scholes. The area of real options applies the option pricing models initially designed for pricing financial options to the area of real investments (i.e. investment projects). Many types of compensation plans can be seen as option-like instruments, and can be evaluated using the methods discussed in this paper.

Appendix A

Definition. (The bivariate negatively skewed lognormal distribution)

Let the two-dimensional random variable (W, S) have the joint probability density function:

$$h(W,S) = \frac{1}{2\pi\sigma_w\sigma\sqrt{1-\rho^2}(\theta-S)(\tau-W)} exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{ln(\theta-S)-\mu}{\sigma}\right)^2 - 2\rho\frac{ln(\theta-S)-\mu}{\sigma}\frac{ln(\tau-W)-\mu_w}{\sigma_w} + \left(\frac{ln(\tau-W)-\mu_w}{\sigma_w}\right)^2\right]\right] \quad (24)$$

for $W < \tau$, $S < \theta$, where σ_w, σ, μ_w, μ, ρ, θ and τ are constants such that $-1 < \rho < 1$, $0 < \sigma_w$, $0 < \sigma$, $-\infty < \mu_w < \infty$, $-\infty < \mu < \infty$, $-\infty < \theta < \infty$, and $-\infty < \tau < \infty$. Then the random variable (W, S) is defined to have a bivariate negatively skewed or upper bound lognormal distribution.

First, it is shown that the function actually represents a density by showing that its integral over the whole plane is 1; that is:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(W,S)\,dS\,dW = 1 \quad (25)$$

The next substitutions are made to simplify the integral:

$$u = \frac{ln(\tau-W)-\mu_w}{\sigma_w}$$

$$v = \frac{ln(\theta-S)-\mu}{\sigma}$$

so that it becomes:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-p^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left[u^2-2\rho uv+v^2\right]\right] dv\,du$$

On completing the square on u in the exponent, it obtains:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left[(u-\rho v)^2+(1-\rho^2)v^2\right]\right] dv\,du$$

And substituting:

$$x = \frac{u-\rho v}{\sqrt{1-\rho^2}}$$

$$dx = \frac{du}{\sqrt{1-\rho^2}}$$

the integral may be written as the product of two simple integrals:

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dv$$

both of which are 1. Equation (25) is thus verified.

Lemma A1. (The marginal distribution of wealth)

If (W, S) has a bivariate negatively skew lognormal distribution, with joint density function given by equation (24), then the marginal distribution of W is a univariate negatively skewed lognormal distribution; that is, $W \sim \Lambda^P(\mu_w, \sigma_w, \tau)$.

Proof: The marginal density of W is, by definition, given by the following equation:

$$h(W) = \int_{-\infty}^{\infty} h(W,S)\,dS$$

and substituting:

$$v = \frac{ln(\theta - S) - \mu}{\sigma},$$

and completing the square on v, yields the following equation:

$$h(W) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_w\sqrt{1-\rho^2}(\tau - W)} exp\left[-\frac{1}{2}\left(\frac{ln(\tau - W) - \mu_w}{\sigma_w}\right)^2 - \frac{1}{2(1-\rho^2)}\left(v - \rho\frac{ln(\tau - W) - \mu_w}{\sigma_w}\right)^2\right] dv$$

Then the substitutions:

$$x = \frac{v - \rho(ln(\tau - W) - \mu_w)/\sigma_w}{\sqrt{1-\rho^2}}$$

$$dx = \frac{dv}{\sqrt{1-\rho^2}}$$

show that:

$$h(W) = \frac{1}{\sqrt{2\pi}\sigma_w(\tau - W)} exp\left[-\frac{1}{2}\left(\frac{ln(\tau - W) - \mu_w}{\sigma_w}\right)^2\right]$$

which is a univariate upper bound or negatively skewed lognormal density.

Lemma A2. (The marginal distribution of the underlying)

If (W,S) has a bivariate negatively skew lognormal distribution, with joint density function given by equation (24), then the marginal distribution of S is a univariate negatively skewed lognormal distribution; that is, $S \sim \Lambda^P(\mu, \sigma, \theta)$.

Proof: The marginal density of S is by definition:

$$h(S) = \int_{-\infty}^{\infty} h(W,S)\,dW$$

and substituting:

$$v = \frac{ln(W - \tau) - \mu_w}{\sigma_w}$$

and completing the square on v, one finds that:

$$h(S) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma\sqrt{1-\rho^2}(\theta - S)} exp\left[-\frac{1}{2}\left(\frac{ln(\theta - S) - \mu}{\sigma}\right)^2 - \frac{1}{2(1-\rho^2)}\left(v - \rho\frac{ln(\theta - S) - \mu}{\sigma}\right)^2\right] dv$$

Then the substitutions:

$$x = \frac{v - \rho(ln(\theta - S) - \mu)/\sigma}{\sqrt{1-\rho^2}}$$

$$dx = \frac{dv}{\sqrt{1-\rho^2}}$$

show that:

$$h(S) = \frac{1}{\sqrt{2\pi}\sigma(\theta - S)} exp\left[-\frac{1}{2}\left(\frac{ln(\theta - S) - \mu}{\sigma}\right)^2\right]$$

which is a univariate upper bound or negatively skewed lgnormal distribution.

Lemma A3. (The conditional distribution of wealth given the underlying)

If (W,S) has a bivariate negatively skewed lognormal distribution, with joint density function given by equation (24), then the conditional distribution of W given $S = s$ is a univariate negatively skewed lognormal distribution; that is $W|S$ is:

$$\Lambda^P\left[\mu_w + \rho\frac{\sigma_w}{\sigma}(ln(\theta - S) - \mu), (\sigma_w^2(1-\rho^2))^{\frac{1}{2}}, \tau\right]$$

Proof: The conditional density of W for fixed values of S is:

$$h(W|S) = \frac{h(S,W)}{h(S)}$$

and after substituting, the expression might be put in the form:

$$h(W|S) = \frac{1}{\sqrt{2\pi}\sigma_w \sqrt{1-\rho^2}(\tau - W)}$$

$$exp\left[-\frac{1}{2\sigma_w^2(1-\rho^2)}[ln(\tau - W)\right.$$

$$\left.-(\mu_w + \rho\frac{\sigma_w}{\sigma}(ln(\theta - S) - \mu))\right]^2\Bigg]$$

which is a univariate upper bound or negatively skewed lognormal density.

Appendix B

Lemma:

Suppose that the stock price Su follows the SDE given by:

$$dS_t = \left[r\theta e^{-r(T-t)} - \left(\theta e^{-r(T-t)} - S_t\right)\mu\right]dt$$

$$-\sigma\left[\theta e^{-r(T-t)} - S_t\right]dB_t \quad (26)$$

where $-\infty < \mu < \infty$, $\sigma > 0$, $-\infty < \theta < \infty$, $r > 0$, and $\theta e^{-rT} > S_0$ are constants, and $t \in E[0, T]$.

Then the solution to this SDE with $t = T$ is given by:

$$S_T = \theta - \left(\theta e^{-rT} - S_0\right)e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma B_T} \quad (27)$$

Proof: (The proof follows Mao (1997), p. 98-99.)

Equation (26) is a scalar linear SDE. The corresponding homogeneous linear equation is the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (28)$$

Then the fundamental solution of equation (28) is given by:

$$\Phi(t) = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t} \quad (29)$$

The explicit solution of equation (26) is given by:

$$S(t) = \Phi(t)\Bigg(S_0 + \int_0^t \Phi^{-1}(u)\left[(r-\mu)\theta e^{-r(T-u)}\right.$$

$$\left.+\theta\sigma^2 e^{-r(T-u)}\right]du - \int_0^t \Phi^{-1}(u)\theta\sigma e^{-r(T-u)}dB_u\Bigg)$$

Substituting equation (29) into the previous equation, and simplifying, yields:

$$S(t) = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}S_0 + \theta e^{-rT}\left(r - \mu + \sigma^2\right)$$

$$\int_0^t e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma B_u}du$$

$$-\sigma\int_0^t e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma B_u}dB_u \quad (30)$$

Now consider the function:

$$V(x, u) = e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma x_u}$$

Then, by applying Itos formula, we obtain the following equality:

$$d\left[e^{r - \mu + \frac{\sigma^2}{2}u - \sigma B_u}\right]$$

$$= \left[r - \mu + \sigma^2\right]e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma B_u}du$$

$$-\sigma e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma B_u}dB_u \quad (31)$$

Equation (31) can be rewritten as:

$$e^{\left(r - \mu + \frac{\sigma^2}{2}\right)t - \sigma B_t} - 1$$

$$= (r - \mu + \sigma^2)\int_0^t e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma B_u}du$$

$$-\sigma\int_0^t e^{\left(r - \mu + \frac{\sigma^2}{2}\right)u - \sigma B_u}dB_u \quad (32)$$

Substituting equation (32) into equation (30) yields, after simplification, the following equation:

$$S_t = \theta e^{-r(T-t)} - \left(\theta e^{-rT} - S_0\right)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t} \quad (33)$$

which yields the desired result at time t = T as given by equation (27).

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